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A strategic mathematical model of waste disposal

Un modelo matemático estratégico de la disposición de residuos

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ABSTRACT

The impact of human actions on the environment poses several challenges at the global level, with solutions that deserve the productive and political sectors and civil society to be jointly involved. Resource scarcity, ecosystem degradation, and climate change must be addressed urgently; this is a paradigm change in the forms of production and consumption. This is a transition from a linear economy to a circular economy, which allows responsible disposal and reuse of waste along the links of production and use. However, despite notable advances in recycling and upcycling, landfill and dump disposal remain the primary waste disposal worldwide. Furthermore, a significant amount of waste is disposed of illegally, affecting the quality of life of communities that live in nearby areas. This work studies the trade-off between waste container removal and illegal micro-dumps cleaning using impulsive control. A type of strategic mathematical model is formulated, one that captures the minimal but relevant aspects of the phenomenon, to describe the dynamics of garbage.

Keywords:

Circular economy, Waste containers, Illegal dump, Impulsive control, Security factor.

RESUMEN

El impacto de las acciones humanas sobre el medio ambiente plantea varios desafíos a nivel global con soluciones que merecen la participación conjunta de los sectores productivos, políticos y de la sociedad civil. La escasez de recursos, la degradación de los ecosistemas y el cambio climático deben ser abordados con urgencia, siendo necesario un cambio de paradigma en las formas de producción y consumo. Sin embargo, a pesar de los notables avances en el reciclaje y el suprareciclaje, los rellenso sanitarios y vertederos siguen siendo la principal forma de eliminación de residuos en todo el mundo. Es más, una cantidad importante de residuos se elimina de manera ilegal afectando la calidad de vida de las comunidades que viven en zonas cercanas. Este trabajo estudia la compensación entre el retiro de contenedores de residuos y la limpieza de micro-vertederos ilegales mediante el control impulsivo. Se formula un modelo matemático de tipo estratégico, aquel que capta aspectos mínimos pero relevantes del fenómeno, para describir la dinámica de la basura.

Palabras Claves:

Economía circular, Contenedores de residuos, Vertederos ilegales, Control impulsivo, Factor de seguridad.

2020 AMS Mathematics Subject Classification: Primary: 92B05; Secondary: 91D10

1 INTRODUCCIÓN

Waste is generated by human action and as a subproduct of satisfying consumer and production needs. This fact is structured by the linear economy, a model that is defined from the chain: take, make, use, and destroy (Ghisellini *et al.*, 2016). Indeed, the quantity of municipal solid waste (MSW) is increasing worldwide as human societies move toward an urban future. Recent estimates suggested that 1.3 billion tonnes of MSW are generated each year; however, this quantity is projected to increase to around 2.2 billion in 2025 (Hoornweg and Bhada-Tata, 2012). As in most countries, in Chile waste is disposed of in legal and illegal dumps, corresponding to 69.4% and 30.6%, respectively (Ministerio de Medio Ambiente, 2022). Although these final disposal places are far from the central area of the city, they represent a sociosanitary problem for the populations living in adjacent areas, such as rural and urban communities. Indeed, residents living close to waste disposal places are typically affected by contamination of water and ground, jointly with bad smells, visual displeasure, and the potential propagation of diseases and plagues (Cárdenas *et al.*, 2016; Ossio and Faúndez, 2021; Escobar, 2021). Eradicating this vulnerability is the long-term seventh goal of Chile's transition to the circular economy 2040, termed "Recovery of sites affected by illegal waste disposal" (Ministerio de Medio Ambiente, 2021), for this reason, it is important to understand the dynamics of waste disposal and removal to detect critical control points and optimize the local resources for its management.

To achieve this important paradigm change, a challenge of the utmost urgency at the global level, it is necessary to work together with the productive and political sectors and civil society (Govindan and Hasanagic, 2018; Ossio *et al.*, 2020; Ministerio de Medio Ambiente, 2021) to avoid the several negative effects that waste disposal generates on the environmental, social and economic dimensions. In this regard, four axes have been outlined to achieve the desired transformation: Circular Innovation, Circular Culture, Circular Regulation, and Circular Territories in relation to each dimension and their interrelation through the government and/or municipal council directions (Ministerio de Medio Ambiente, 2021).

Since it is a transversal problem, illegal disposal of waste aggravates environmental, social, and economic impacts. For example, Vergara and Tchobanoglous (2012) showed that, relative to the areas surrounding dumping sites, stream ecology, flora and fauna, habitat depletion, and land use change dominated the concerns of the stakeholders. This contingency is not unfamiliar among Chilean communities; indeed, according to Ossio and Faúndez (2021), in Chile, there are 3.735 illegal sites of final waste disposal formed by 3.492 dumps and 243 micro-dumps, distributed throughout the country; however, these are concentrated in the Región Metropolitana which generates the largest amount of waste (Ministerio de Medio Ambiente, 2022; Vivanco Font, 2023). Several strategies have been proposed to recover sites affected by illegal waste disposal, such as zero waste industries, educating the civil society to promote recycling/upcycling behaviors, and strengthening control to avoid illegal waste disposal (Ministerio de Medio Ambiente, 2021).

In this work, a strategic mathematical model (Jiliberto, 2020) is formulated to describe the dynamics of waste that is deposited both in legal waste containers and littering in clandestine or illegal micro-dumps. Assuming that waste container removal and illegal micro-dump cleaning are carried out simultaneously and regularly, a trade-off occurs. In addition, depending on the rate at which wastes are littered, the waste containers can collapse, and thus the waste in the illegal micro-dumps increases and maintains. This situation is modeled from the level of filling waste containers, a fraction between availability and occupied capacity.

2 MATHEMATICAL MODELING

Let be $G = G(t)$ the total waste at time $t \geq 0$. Assuming that the waste is deposited by individuals on the municipal council waste containers or sites such as hillsides and rural roads on the periphery of the city giving rise to micro-dumps, their amounts are represented respectively by $G_\oplus = G_\oplus(t)$ and $G_\ominus = G_\ominus(t)$. Then, $G = G_\oplus + G_\ominus$. In addition, the total capacity of municipal council waste containers is given by the density of these, represented as $N = N(t)$, and their specific capacity c. Therefore, when $G_\oplus = cN$ is obtained, the municipal council waste containers are filled. However, since this notion can be subjective, the occupancy fraction $\gamma \in (0,1)$ equivalent to $G_\oplus/cN$ is proposed. Consequently, when the municipal council waste containers have availability, being fulfilled $G_\oplus < \gamma cN$, then these are occupied at a rate proportional to available capacity: $r(1 - G_\oplus/cN)$. Conversely, when $G_\oplus \geq \gamma cN$ is obtained, the inflow waste that cannot be deposited due to lack of space is dumped, in addition to rate $r_\ominus$, in peripheral sites, and then, forms the illegal micro-dumps. In turn, considering a population increase or constant container theft at a rate μ, the density of municipal waste containers decreases. This waste dynamics' occurs each $\tau > 0$ unit of time, in concordance with the municipal council waste container removal by the cleaning and maintenance department. At these moments, a fraction $\lambda = 1 - e^{-\mu\tau}$ of stolen waste containers are replenished, jointly with the cleaning of illegal micro-dumps and the container waste removal in a trade-off fractions $\delta \in (0,1)$ and $1 - \delta$, respectively. In fact, municipal council waste containers are removed and their capacity is restored to cN when $\delta = 1$. However, the illegal micro-dumps were not cleaned. Conversely, when $\delta = 0$, the illegal micro-dumps were cleaned but the waste containers were not removed, which promotes the emergence of illegal micro-dumps. Consequently, a trade-off is obtained between these clean spaces. Therefore, the following mathematical model is proposed:

$$\left\{\begin{array}{lcl} N'(t) &=& -\mu N(t) \\ G'_{\oplus}(t) &=& r_{\oplus}\left(1-\dfrac{G_{\oplus}(t)}{cN(t)}\right) \\ G'_{\ominus}(t) &=& \left\{\begin{array}{ll} r_{\ominus} & ,\ \text{if}\quad G_{\oplus}(t)<\gamma cN(t) \\ r_{\ominus}+r_{\oplus}\dfrac{G_{\oplus}(t)}{cN(t)} & ,\ \text{if}\quad G_{\oplus}(t)\geq\gamma cN(t) \end{array}\right. \end{array}\right\} t\neq k\tau \quad (1)$$
$$\left.\begin{array}{lcl} N(t^{+}) &=& N(t)+(1-e^{-\mu\tau})(N_{*}-N(t)) \\ G_{\oplus}(t^{+}) &=& (1-\delta)G_{\oplus}(t) \\ G_{\ominus}(t^{+}) &=& \delta G_{\ominus}(t) \end{array}\right\} t=k\tau$$

with $N(0)=N_{*}$, $G_{\ominus}(0)>0$, and $G_{\oplus}(0)>0$. Importantly, if $\mu=0$ then $N(t)=N_{*}$ for any $t\geq 0$.

3 RESULTS

The mathematical study of the model (1), described by a system of impulsive differential equations, investigates the long-term patterns of waste dynamics and focuses on the relationship between the instant at which the waste containers are filled and when they are removed, and jointly when the illegal micro-dumps are cleaned. The synchrony between activities plays a key role in the illegal micro-dumps' non-persistence.

THRESHOLD CONDITION AND TEMPORAL DYNAMICS

Let be $\{t_k\}$ an increasing and non-bounded sequence such that $t_{k+1}=t_k+\tau$, which is related to the removal of the waste containers and cleaning the illegal micro-dumps. Then, solving the model (1) for $t\in(t_k,t_{k+1}]$, follows that

$$N(t)=N(t_k^{+})e^{-\mu(t-t_k)}.$$

Taking $t=t_{k+1}$, the stroboscopic map

$$N(t_{k+1})=[N(t_k)+(1-e^{-\mu\tau})(N_{*}-N(t_k))]e^{-\mu\tau}$$

is obtained, whose equilibrium point is given by

$$\overline{N}=\frac{\lambda}{e^{\mu\tau}-1+\lambda}N_{*},$$

where $\lambda=1-e^{-\mu\tau}$. Consequently,

$$G'_{\oplus}(t)+\frac{r_{\oplus}}{cN(t_k^{+})e^{-\mu(t-t_k)}}G_{\oplus}(t)=r_{\oplus},$$

has by solution

$$G_{\oplus}(t)=\exp\left\{-\frac{r_{\oplus}(e^{\mu(t-t_k)}-1)}{cN(t_k^{+})\mu}\right\}\left(G_{\oplus}(t_k^{+})+\int_{t_k}^{t}r_{\oplus}E(s)ds\right),$$

where

$$E(s)=\exp\left\{\frac{r_{\oplus}(e^{\mu(s-t_k)}-1)}{cN(t_k^{+})\mu}\right\}$$

for any $t\in(t_k,t_{k+1}]$. Taking $t=t_{k+1}$, we have the stroboscopic map

$$G_{\oplus}(t_{k+1})=(1-\delta)\exp\left\{-\frac{r_{\oplus}(e^{\mu\tau}-1)}{cN(t_k^{+})\mu}\right\}G_{\oplus}(t_k)+$$
$$\underbrace{+\exp\left\{-\frac{r_{\oplus}(e^{\mu\tau}-1)}{cN(t_k^{+})\mu}\right\}\int_{t_k}^{t_{k+1}}r_{\oplus}\exp\left\{\frac{r_{\oplus}(e^{\mu(s-t_k)}-1)}{cN(t_k^{+})\mu}\right\}ds}_{\mathscr{I}},$$

whose equilibrium point is given by

$$\overline{G}_{\oplus}=\frac{r_{\oplus}\exp\left\{-\dfrac{r_{\oplus}(1-e^{-\mu\tau})}{c\overline{N}\mu}\right\}}{1-(1-\delta)\exp\left\{-\dfrac{r_{\oplus}(1-e^{-\mu\tau})}{c\overline{N}\mu}\right\}}\cdot\mathscr{L},$$

where $\mathscr{L}$ exist due to $0<\mathscr{I}\leq r_{\oplus}\tau$.

Taking $e^{\mu(t-t_k)}>\mu(t-t_k)+1$ for any $t\in(t_k,t_{k+1}]$, follows that

$$E(t)>\exp\left\{\frac{r_{\oplus}(t-t_k)}{cN(t_k^{+})}\right\},$$

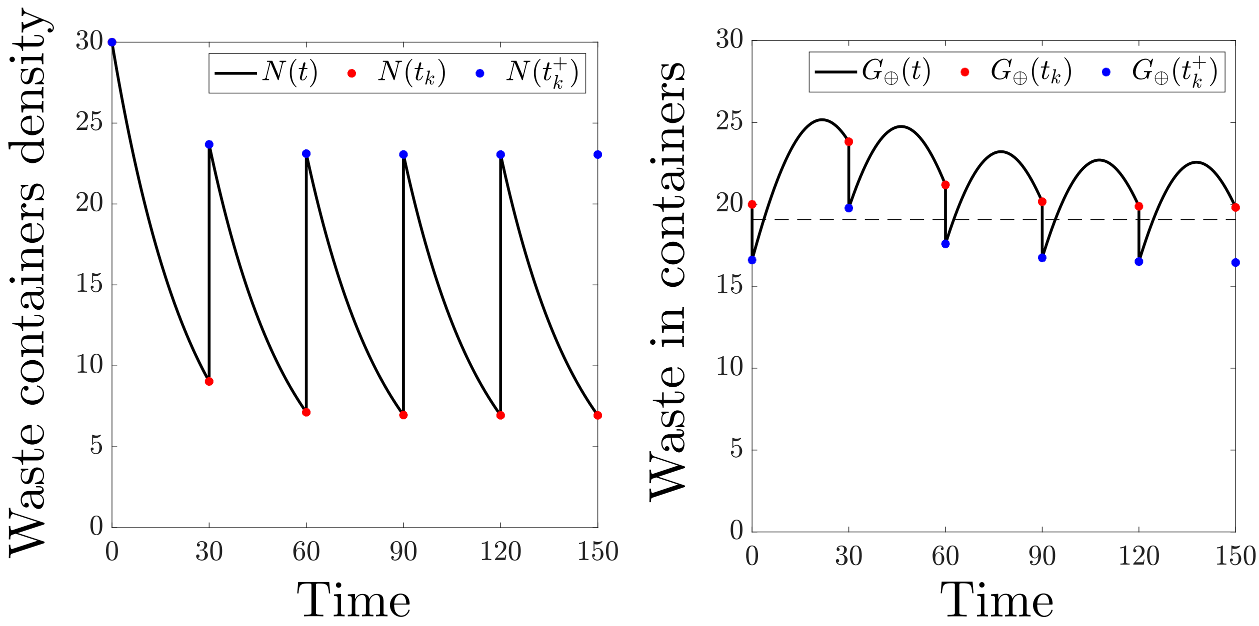
and then

$$\begin{aligned} \mathscr{L} &\geq \lim_{k\to\infty}\frac{cN(t_k^{+})}{r_{\oplus}}\left(\exp\left\{\frac{r_{\oplus}\tau}{cN(t_k^{+})}\right\}-1\right), \\ &= \frac{c\overline{N}e^{\mu\tau}}{r_{\oplus}}\left(\exp\left\{\frac{r_{\oplus}\tau}{c\overline{N}e^{\mu\tau}}\right\}-1\right) \end{aligned} \quad (2)$$

is obtained. Figure 1 illustrates the temporal dynamics of model (1) for $N(t)$ and $G_{\oplus}(t)$ states and the respective stroboscopic maps according to impulsive dynamics.

Figure 2 shows that $d\overline{G}_{\oplus}/d\mu<0$, and thus $\overline{G}_{\oplus}$ tends to

$$\overline{G}_{\oplus *}=\frac{cN_{*}\left(1-\exp\left\{-\dfrac{r_{\oplus}\tau}{cN_{*}}\right\}\right)}{1-(1-\delta)\exp\left\{-\dfrac{r_{\oplus}\tau}{cN_{*}}\right\}} \quad (3)$$



(a) (b)

Figure 1: Temporal dynamics of model (1). (a) The waste containers density, $N = N(t)$, and (b) the amount of waste in containers, $G_\oplus = G_\oplus(t)$ for $t \in [0,150]$ and initial conditions $N(0) = 30$, $G_\oplus(0) = 20$. The common parameter values are $\tau = 30$, $\mu = 0.04$, $c = 2$, $\delta = 0.17$, $\gamma = 0.95$, $r_\ominus = 0.01$, $r_\oplus = 1$, and $N_* = 30$. Importantly, the dashed line represents the top-fill level, $G_\oplus := \gamma c N_* = 19$.

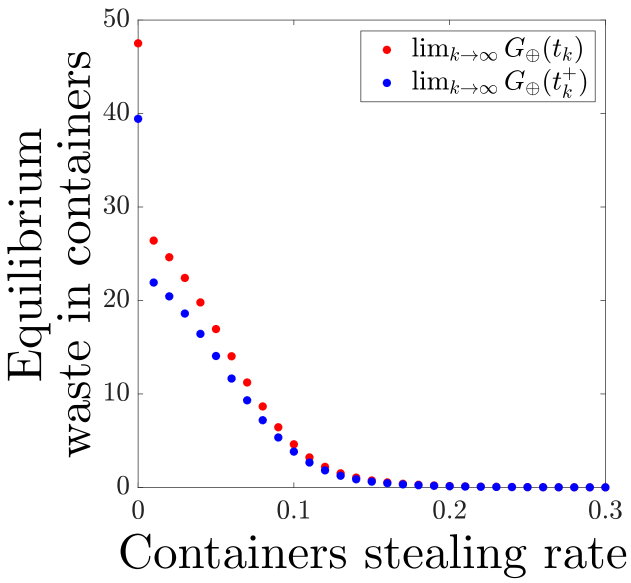


Figure 2: Equilibrium values of the stroboscopic maps $G_\oplus(t_{k+1}) = F(G_\oplus(t_k), N(t_k))$ (red dots) and $G_\oplus(t_{k+1}^+) = \delta G_\oplus(t_{k+1})$ (blue dots) as μ increases, using $\tau = 30$, $c = 2$, $\delta = 0.17$, $\gamma = 0.95$, $r_\ominus = 0.01$, $r_\oplus = 1$, and $N_* = 30$ as parameter values, and initial conditions $N(0) = 30$, $G_\oplus(0) = 20$.

as μ tends to zero, and thus necessarily, $\mathscr{L}$ tends to $cN_*(\exp\{r_\oplus \tau/(cN_*)\} - 1)/r_\oplus$ according to (2).

On the other hand, let be $\{s_k\}$ a sequence such that $G_\oplus(s_k) = \gamma c N_*$, which is related to the waste containers are filled. Therefore, integrating the model (1) on $(s_k, s_{k+1}]$ we have

$$G_\oplus(s_{k+1}) = cN_* + [cN_* - G_\oplus(s_k^+)]\exp\left\{-\frac{r_\oplus(s_{k+1} - s_k)}{cN_*}\right\}.$$

Assuming that $G_\oplus(s_k^+) = 0$, this is, the waste containers are removals, follows that

$$\gamma c N_* = cN_*\left(1 - \exp\left\{-\frac{r_\oplus(s_{k+1} - s_k)}{cN_*}\right\}\right)$$

and thus,

$$s_{k+1} = s_k + \underbrace{\frac{cN_*}{r_\oplus}\ln\left(\frac{1}{1-\gamma}\right)}_{T},$$

which is an increasing and non-bounded sequence. This scenario is the particular case of the model (1) when $\delta = 0$ is proposed.

Therefore, the mathematical model (1) provides a theoretical framework that captures minimal but relevant aspects of the waste dynamics associated with the cleaning of illegal micro-dumps and removal of waste containers based on impulsive control. Specifically, we can derive the following conclusion.

Proposition 1 *Let be* $\mu = 0$ *and*

$$\mathscr{R} = \frac{\delta}{(1-\gamma)\left[\delta + \exp\left\{\frac{r_\oplus \tau}{cN_*}\right\} - 1\right]}.$$

Therefore,

$$G_\oplus(t) = cN_* - [cN_* - (1-\delta)G_\oplus(t_k)]\exp\left\{-\frac{r_\oplus(t-t_k)}{cN_*}\right\}$$

satisfies the model (1) for any $t \in (t_k, t_{k+1}]$, *where*

$$G_\oplus(t_{k+1}) = (1-\delta)\exp\left\{-\frac{r_\oplus\tau}{cN_*}\right\}G_\oplus(t_k) + cN_*\left(1-\exp\left\{-\frac{r_\oplus\tau}{cN_*}\right\}\right)$$

and $k \geq 0$, *so that* $G_\oplus(t) \in [(1-\delta)\overline{G}_{\oplus *}, \overline{G}_{\oplus *}]$ *as t tends to infinity. In addition, if* $\mathscr{R} \geq 1$ *then*

$$G_\ominus(t) \in [r_\ominus\tau\delta/(1-\delta), r_\ominus\tau/(1-\delta)]$$

as t tends to infinity. Conversely, when $0 < \mathscr{R} < 1$ *follows that*

$$G_\ominus(t) \in [\delta\overline{G}_{\ominus *}, \overline{G}_{\ominus *}],$$

where $\overline{G}_{\ominus *}$ *is given by (8) when* $(1-\delta)G_\oplus(t_k) < cN_*$, *and by (9) when* $(1-\delta)G_\oplus(t_k) \geq cN_*$ *are obtained from* $k \geq k_*$.

Proof

Let be $\{t_k\}$ and $\{s_k\}$ two sequences given by that $t_{k+1} = t_k + \tau$ and $s_{k+1} = s_k + T_k$ such that $G_\oplus(s_k) = \gamma cN_*$. Integrating on $(t_k, t_{k+1}]$ follows that

$$G_\oplus(t) = cN_* - [cN_* - G_\oplus(t_k^+)]\exp\left\{-\frac{r_\oplus(t-t_k)}{cN_*}\right\}, \quad (4)$$

which solve the model (1). Thus, taking $t = t_{k+1}$ we have the stroboscopic map

$$G_\oplus(t_{k+1}) = (1-\delta)\exp\left\{-\frac{r_\oplus\tau}{cN_*}\right\}G_\oplus(t_k) + cN_*\left(1-\exp\left\{-\frac{r_\oplus\tau}{cN_*}\right\}\right). \quad (5)$$

Analogously, for $s \in (s_k, s_{k+1}]$,

$$G_\oplus(s_{k+1}) = cN_* - [cN_* - G_\oplus(s_k^+)]\exp\left\{-\frac{r_\oplus T_k}{cN_*}\right\},$$

is obtained, and equivalent to

$$\gamma cN_* = cN_* - [cN_* - G_\oplus(s_k^+)]\exp\left\{-\frac{r_\oplus T_k}{cN_*}\right\},$$

or

$$G_\oplus(s_{k+1}) = cN_* - [cN_* - \gamma cN_*]\exp\left\{-\frac{r_\oplus T_k}{cN_*}\right\}$$

depending on the initial condition value. Therefore,

$$G_\oplus(s_k^+) = cN_*\left(1-(1-\gamma)\exp\left\{\frac{r_\oplus T_k}{cN_*}\right\}\right)$$

with $0 < T_k \leq T$, and

$$G_\oplus(s_{k+1}) = cN_*\left(1-(1-\gamma)\exp\left\{-\frac{r_\oplus T_k}{cN_*}\right\}\right)$$

with $T_k \geq T$, are obtained.

Considering the equilibrium point of sequence $\{G_\oplus(t_k)\}_k$, given by the expression (3), it is necessary to study the parametric conditions that allow it to occur that $\overline{G}_{\oplus *} > G_\oplus(s_k^+)$ so that the waste containers to be removals before these are full, and $\overline{G}_{\oplus *} > G_\oplus(s_{k+1})$ in the collapse case.

Firstly, $\overline{G}_{\oplus *} > G_\oplus(s_k^+)$ is equivalent to

$$\frac{cN_*\left(1-\exp\left\{-\frac{r_\oplus\tau}{cN_*}\right\}\right)}{1-(1-\delta)\exp\left\{-\frac{r_\oplus\tau}{cN_*}\right\}} > N_*\left(1-(1-\gamma)\exp\left\{\frac{r_\oplus T_k}{cN_*}\right\}\right),$$

if and only if,

$$\exp\left\{\frac{r_\oplus T_k}{cN_*}\right\} > \frac{\delta}{(1-\gamma)\left[\delta + \exp\left\{\frac{r_\oplus\tau}{cN_*}\right\} - 1\right]} = \mathscr{R}.$$

Solving for T_k, a value exists whether $\mathscr{R} > 1$. Therefore, $G'_\ominus(t) = r_\ominus$ for any $t \neq t_k$ and $G_\ominus(t_k^+) = \delta G_\ominus(t_k)$, so that $G_\ominus(t) = \delta G_\ominus(t_k) + r_\ominus(t-t_k)$ for any $t \in (t_k, t_{k+1}]$, where the sequence $\{G_\oplus(t_k)\}_k$ is increasing and satisfies $G_\ominus(t_{k+1}) = \delta G_\ominus(t_k) + r_\ominus\tau$. Consequently, the solution of this difference equation is given by

$$G_\ominus(t_k) = \delta^k G_\ominus(t_0) + r_\ominus\tau \cdot \frac{1-\delta^k}{1-\delta} \quad (6)$$

with tends to $r_\ominus/(1-\delta)$ as k increases.

Secondly, from $\overline{G}_{\oplus *} > G_\oplus(s_{k+1})$ and by procedures analogous to the first case, $\exp\{-r_\oplus T_k/cN_*\} > \mathscr{R}$ is obtained. Solving for T_k, a value exists whether $0 < \mathscr{R} < 1$. Using the expression (4) at $t = s_k$ follows that

$$\gamma cN_* = cN_* - [cN_* - (1-\delta)G_\oplus(t_k)]\exp\left\{-\frac{r_\oplus(s_k-t_k)}{cN_*}\right\},$$

if and only if,

$$s_k = t_k + \frac{cN_*}{r_\oplus}\ln\left(\frac{cN_* - (1-\delta)G_\oplus(t_k)}{cN_*(1-\gamma)}\right) \quad (7)$$

when $(1-\delta)G_\oplus(t_k) < cN_*$ from $k \geq k_*$. Importantly,

$$\begin{aligned} T_k &= s_{k+1} - s_k, \\ &= \tau + \frac{cN_*}{r_\oplus}\ln\left(\frac{cN_* - (1-\delta)G_\oplus(t_{k+1})}{cN_* - (1-\delta)G_\oplus(t_k)}\right) \end{aligned}$$

tends to τ as k increases. Therefore,

$$G_\ominus(t) = \begin{cases} \delta G_\ominus(t_k) + r_\ominus(t-t_k) & , \quad \text{If} \quad t \in (t_k, s_k] \\ \tilde{G}_\ominus(t) & , \quad \text{If} \quad t \in [s_k, t_{k+1}] \end{cases}$$

where

$$\tilde{G}_{\ominus}(t) = G_{\ominus}(s_k) + (r_{\ominus} + r_{\oplus})(t - s_k) - \\ - [cN_* - (1-\delta)G_{\oplus}(t_k)] \left(\exp\left\{ -\frac{r_{\oplus}(s_k - t_k)}{cN_*} \right\} - \\ - \exp\left\{ -\frac{r_{\oplus}(t - t_k)}{cN_*} \right\} \right).$$

is obtained. Taking $t = s_k$ and $t = t_{k+1}$, we have

$$G_{\ominus}(s_k) = r_{\ominus}(s_k - t_k) + \delta G_{\ominus}(t_k),$$

and

$$G_{\ominus}(t_{k+1}) = G_{\ominus}(s_k) + (r_{\ominus} + r_{\oplus})(\tau - (s_k - t_k)) \\ - [cN_* - (1-\delta)G_{\oplus}(t_k)] \left(\exp\left\{ -\frac{r_{\oplus}(s_k - t_k)}{cN_*} \right\} - \\ - \exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\} \right).$$

Therefore, substituting the first equation into the second equation, and considering the difference $s_k - t_k$, given by (7), we have

$$G_{\ominus}(t_{k+1}) = \delta G_{\ominus}(t_k) + (r_{\oplus} + r_{\ominus})\tau - \\ cN_* \ln\left(\frac{cN_* - (1-\delta)G_{\oplus}(t_k)}{cN_*(1-\gamma)} \right) - cN_*(1-\gamma) + \\ + [cN_* - (1-\delta)G_{\oplus}(t_k)] \exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\},$$

whose equilibrium point is given by

$$\overline{G}_{\ominus *} = \frac{(r_{\oplus} + r_{\ominus})\tau - cN_* \ln\left(\frac{cN_* - (1-\delta)\overline{G}_{\oplus *}}{cN_*(1-\gamma)} \right) - cN_*(1-\gamma) + + [cN_* - (1-\delta)\overline{G}_{\oplus *}] \exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\}}{1-\delta}. \tag{8}$$

On the other hand, if $(1-\delta)G_{\oplus}(t_k) \geq cN_*$ from $k \geq k_*$ then, integrating on $(t_k, t_{k+1}]$ we have

$$G_{\ominus}(t) = G_{\ominus}(t_k) + (r_{\ominus} + r_{\oplus})(t - t_k) + \\ + [(1-\delta)G_{\oplus}(t_k) - cN_*] \exp\left\{ -\frac{r_{\oplus}(t - t_k)}{cN_*} \right\},$$

and taking $t = t_{k+1}$, follows that

$$G_{\ominus}(t_{k+1}) = \delta G_{\ominus}(t_k) + (r_{\ominus} + r_{\oplus})\tau + \\ + [(1-\delta)G_{\oplus}(t_k) - cN_*] \exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\},$$

whose equilibrium point is given by

$$\overline{G}_{\ominus *} = \frac{(r_{\oplus} + r_{\ominus})\tau + [(1-\delta)\overline{G}_{\oplus *} - cN_*] \exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\}}{1-\delta}. \tag{9}$$

Therefore, $G_{\ominus}(t_k)$ tends to $\overline{G}_{\ominus *}$ given by (8) or (9) as k increases, according on the fulfillment of $(1-\delta)G_{\oplus}(t_k) < cN_*$ or $(1-\delta)G_{\oplus}(t_k) \geq cN_*$ from $k \geq k_*$.

Finally, if $\mathscr{R} = 1$ then

$$\gamma = 1 - \frac{\delta}{\delta + \exp\left\{ \frac{r_{\oplus}\tau}{cN_*} \right\} - 1} = 1 - \frac{\delta \exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\}}{1 - (1-\delta)\exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\}},$$

if and only if,

$$\gamma cN_* = \frac{cN_* \left(1 - \exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\} \right)}{1 - (1-\delta)\exp\left\{ -\frac{r_{\oplus}\tau}{cN_*} \right\}} = \overline{G}_{\oplus *}.$$

Thus, $G_{\oplus}(s_k) = \overline{G}_{\oplus *}$ and $s_k = t_k$ from $k \geq \tilde{k}$. Therefore,

$$G_{\ominus}(t) = \delta G_{\ominus}(t_k) + r_{\ominus}(t - t_k)$$

for $t \in (t_k, t_{k+1}]$. Using the (6) relationship, it follows the result. □

Regarding the threshold value, denoted by $\mathscr{R}$, using the concept of elasticity (Martcheva, 2015), which is defined by $\varepsilon^p_{\mathscr{R}} := (\partial\mathscr{R}/\partial p) \cdot (p/\mathscr{R}) \approx \%\Delta\mathscr{R}/\%\Delta p$ where p is some parameter of interest, follows that

$$\varepsilon^{\gamma}_{\mathscr{R}} = \frac{\gamma}{1-\gamma}, \quad \varepsilon^{\delta}_{\mathscr{R}} = 1 - \frac{\delta}{\delta + e^{\kappa} - 1}, \quad \varepsilon^{\kappa}_{\mathscr{R}} = -\frac{\kappa e^{\kappa}}{\delta + e^{\kappa} - 1},$$

where $\kappa = r_{\oplus}\tau/\{cN_*\}$ is the maximal occupancy ratio. Thus, $\varepsilon^{\gamma}_{\mathscr{R}} > 0$, $\varepsilon^{\delta}_{\mathscr{R}} > 0$ and $\varepsilon^{\kappa}_{\mathscr{R}} < 0$. Therefore, as the subjective occupancy fraction and the removal fraction increase or the maximal occupancy ratio decreases, the available capacity of waste containers is promoted (see Figure 3). However, increasing the removal fraction increases the waste amount range in micro-dump in the long term (see Proposition 1).

NUMERICAL SIMULATIONS

To validate our mathematical result, Figure 4 illustrates the Proposition 1 conclusions' based on the varying γ parameter value, and thus, in the filling level γcN_*. Consequently, $\mathscr{R}$ and γcN_* values decrease as γ decreases too, and the inequality fulfillment $(1-\delta)G_{\oplus}(t) < cN_*$ transit to $(1-\delta)G_{\oplus}(t) \geq cN_*$, which can also be promoted as δ increases to one. However, this variation increases significantly the waste amount range in the illegal micro-dumps according to the trade-off between removal and cleaning.

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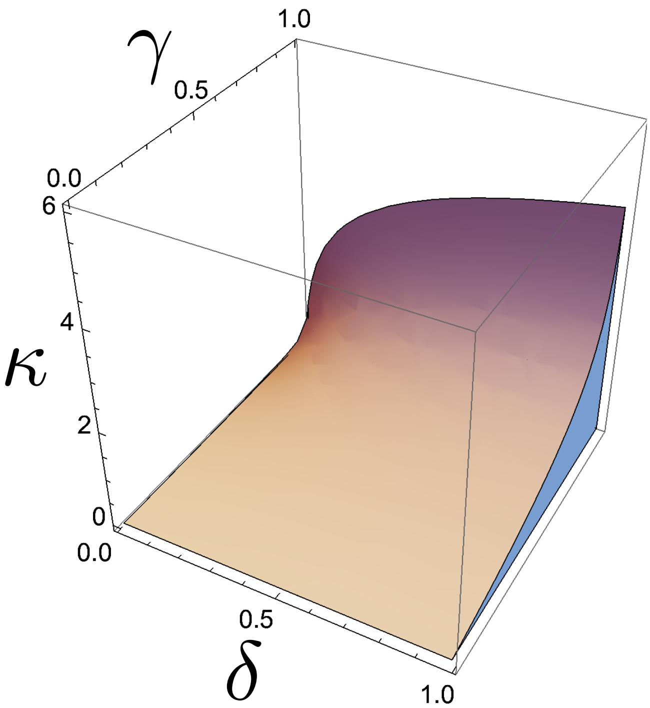


Figure 3: Combinations of $\delta \in (0,1)$, $\gamma \in (0,1)$, and $\kappa \in (0,6)$ so that $\mathscr{R} > 1$, where $\kappa = r_{\oplus}\tau/\{cN_*\}$. Importantly, the complementary region that represents $0 < \mathscr{R} < 1$ is significantly greater than $\mathscr{R} \geq 1$ region.

On the other hand, Figure 5 shows how the temporal dynamics of the model (1) vary as the container stealing rate μ increases, taking as reference the dynamics associated with $\mu = 0$ and a parameter set so that $\mathscr{R} \geq 1$ is obtained (see Fig.5(a)). Here, it is observed the transition among the different scenarios given by Proposition 1, where the capacity of containers in the long term allows a constant waste inflow or not, because these are full, implying waste increases in illegal micro-dumps. Importantly, it is the effect steal of waste containers on garbage dynamics, significantly reducing the containers' top-fill level and promoting their collapse, and thus the illegal micro-dump maintenance.

4 DISCUSSION AND CONCLUSIONS

Our goal was to study the trade-off dynamics between the removal of waste containers and the cleaning of illegal micro-dumps. This dynamics was represented using a mathematical model described by an impulsive differential system at both fixed and variable times (Cordova-Lepe *et al.*, 2015). The findings establish two scenarios in the long term, the containers are full or not, and whose differentiation depends on a threshold that is a function of all model parameters. In particular, when the waste container density does not vary (taking $\mu = 0$), an explicit representation is obtained for this threshold value, denoted by $\mathscr{R}$. Thus, when $0 < \mathscr{R} < 1$ the containers are full in the long term. Conversely, when $\mathscr{R} \geq 1$ the containers have a capacity for waste disposal. Therefore, it is natural to associate $\mathscr{R}$ as a *safety factor* that relates the capacity and demand by a quotient. From the *safety factor* is possible to monitor and guard the integrity of a specific process, particularly of engineering (Hansson, 2009).

Based on elasticity analysis of static quantities (Martcheva, 2015), this *safety factor* increases as the removal (δ) and/or the subjective occupancy (γ) fractions increase too, or by decreasing the maximal occupancy ratio ($\kappa = r_{\oplus}\tau/\{cN_*\}$). As the trade-off result, whose consequence implies that illegal micro-dumps persist, our findings establish that the efforts must be aimed at γ increase or κ decrease. Firstly, we established that γ represents the subjective occupancy fraction, a measure that pretends to consider human behavior faced in the disposition of legal waste disposal, where the location and access to waste containers are keys in the promotion and maintenance of habits with a socio-environmental co-responsibility (Valenzuela-Levi and Flores-Castillo, 2023). Secondly, the decrease of κ, by the reduction of $r_{\oplus}$ or the increase of N_*, is associated with promoting strategies of recycling and upcycling (Ministerio de Medio Ambiente, 2021; Yang *et al.*, 2023; Valenzuela-Levi, 2019).

The illegal disposal of solid waste in urban areas has been found to affect the structure and function of natural ecosystems (Vergara and Tchobanoglous, 2012). As a result, it is crucial to study and simulate waste management practices in cities to create sustainable strategies that can reduce the environmental and health hazards linked with improper waste disposal. In this regard, comprehending the elements contributing to the development of small-scale waste disposal locations and the measures implemented by local government agencies is vital for making informed choices that reduce the occurrence of new dumping sites (Shmelev and Powell, 2006).

Hence, developing models that represent disposal behaviors in these settings is important for devising efficient measures to address the growing issue of municipal solid waste. A future improvement that could be made in this regard might be the incorporation of a spatially explicit analytic framework to the problem of predicting where would be the most probable location of a new micro-dumping area, based on both, the distribution of the collecting containers and the "environmental" characteristics of frequent micro-dumping points.

There are germ or nuclear mathematical models; that is, although they do not fit in detail with each of the observable expressions, that is, the variations of particular phenomena, they come to contain the minimum elements and relationships to characterize the essence of the class in that such phenomena are inserted. This is the case with the logistic model in population dynamics or the SIR (susceptible-infectious-removed) propagation model in epidemiology. These models, called strategic (Jiliberto, 2020), act as platforms since they have the property that, when assembled or added specificity, they reach a resolution that can be contrasted with a specific reality, and, in general, they become *ad hoc* instruments, that is, with greater descriptive and projection possibilities. In our opinion, the model (1) aspires to be strategic, since considering the complexity that waste disposal processes have *in situ*, its minimalist conceptual structure achieves connections whose mathematical analysis interpreted at sight makes practical sense.

Author contributions statement

R.G, F.C-L, and I.S.A-R conceived the study. R.G and F.C-L developed the theoretical formalism. R.G performed the analytic calculations and the numerical simulations. All authors discussed the results and contributed to the final manuscript.

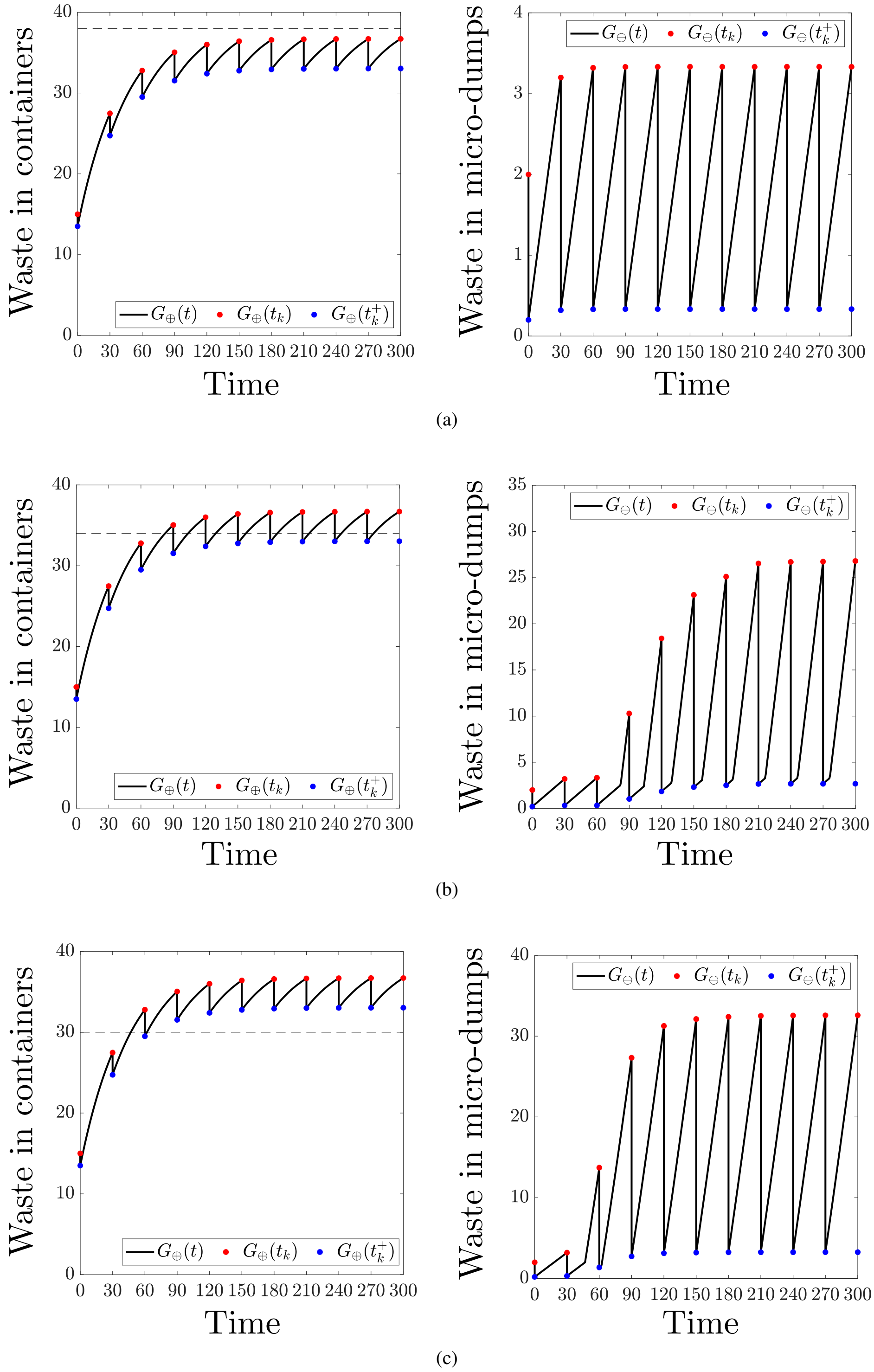
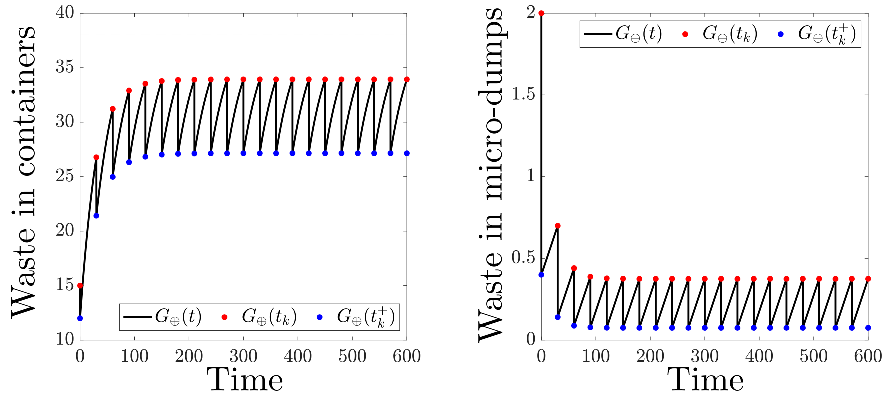
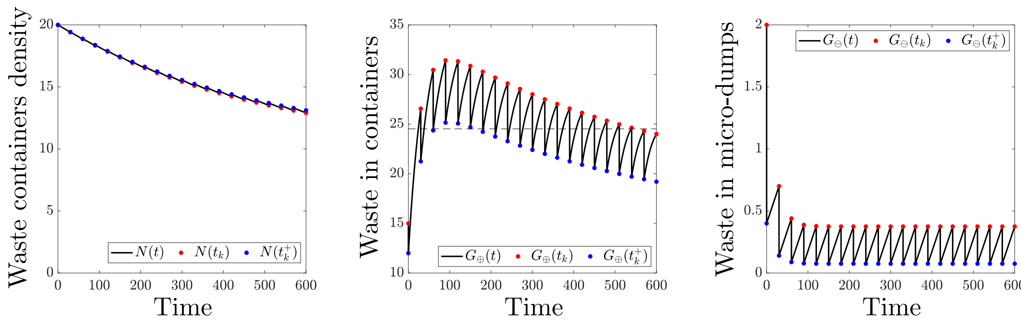


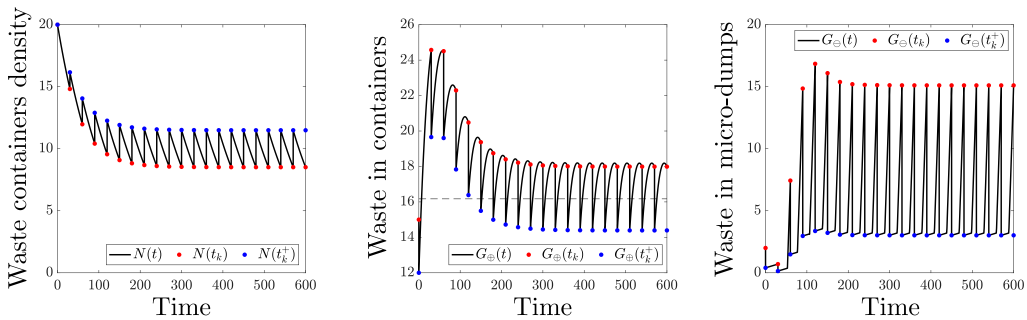
Figure 4: Temporal dynamics of model (1) according to Proposition 1. The common parameter values are $\tau = 30$, $\mu = 0$, $c = 2.0$, $\delta = 0.1$, $r_{\ominus} = 0.1$, $r_{\oplus} = 1.0$, and $N_* = 30$. Particularly, (a) $\gamma = 0.95$ with $\mathscr{R} \approx 1.6434$ so that $G_{\ominus} \in [0.33, 3.33]$, (b) $\gamma = 0.85$ with $\mathscr{R} \approx 0.5478$ so that $G_{\ominus} \in [2.68, 26.81]$, and (c) $\gamma = 0.75$ with $\mathscr{R} \approx 0.3287$ so that $G_{\ominus} \in [3.25, 32.58]$. Importantly, the dashed line represents the top-fill level, $G_{\oplus} = \gamma c N_*$.



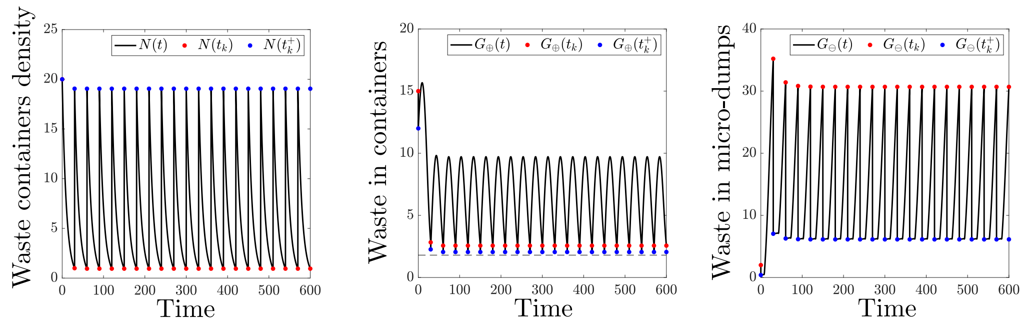
(a)



(b)



(c)



(d)

Figure 5: Temporal dynamics of model (1) as parameter value μ increases. The common parameter values are $\tau = 30$, $c = 2.0$, $\delta = 0.2$, $r_\ominus = 0.01$, $r_\oplus = 1.0$, $\gamma = 0.95$, and $N_* = 20$ so that $\mathscr{R} = 3.0372$. In (a) $\mu = 0$, (b) $\mu = 0.001$, (c) $\mu = 0.01$, and (d) $\mu = 0.1$. Importantly, the dashed line represents the top-fill level, $G_\oplus = \gamma c\bar{N}$.

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